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PLANE STRESS EQUATIONS FOR THE VON MISES–SCHLEICHER YIELD CRITERION

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Systems of equations for plastic stresses and velocities based on the von Mises–Schleicher criterion are obtained for plane stresses. The regions of ellipticity and hyperbolicity of these systems are found, and the limiting stresses and fracture directions identified with the characteristics of the velocity field equations are determined. The results agree well with experimental data for plastic and brittle materials.

Key words: *plasticity, fracture, hyperbolicity, von Mises–Schleicher strength criterion.*

Metal plasticity theory widely uses the Houber–von Mises criterion, according to which the shear–stress intensity in a material takes a constant value upon reaching the yield point. Simple mathematical formulation, energy substantiation, and experimental verification have shown advantages of this criterion over the other criterions for plastic metals [1, 2]. For brittle metals and rocks, however, the Coulomb–Mohr strength criterion [2–8] is the most widely used.

The generalization of this criterion to brittle materials proposed by Schleicher and elaborated later by Nádai states that for plastic flow or fracture of solids, the shear-stress intensity is a certain function of the mean normal stress [1, 2]. For the von Mises–Schleicher criterion, plastic strains can be determined using the associated and nonassociated models [3–6].

For an arbitrary stress state, the von Mises–Schleicher criterion has the form

$$T + \beta\sigma = k, \tag{1}$$

where k is the adhesion, β is the internal-friction coefficient, $\sigma = (\sigma_x + \sigma_y + \sigma_z)/3$ is the mean normal stress, and $T = [(\sigma_x - \sigma_z)^2 + (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + 6\tau_{xy}^2 + 6\tau_{xz}^2 + 6\tau_{yz}^2]^{1/2}/\sqrt{6}$ is the shear-stress intensity.

In the space of the principal normal stresses σ_1, σ_2, and σ_3, the von Mises–Schleicher yield (strength) criterion is interpreted as a circular cone whose vertex lies on the hydrostatic axis. We denote the coordinate of the vertex of the cone by σ_0 ($\sigma_1 = \sigma_2 = \sigma_3 = \sigma_0$). If k and β are constants in the plastic region, they can be determined from results of two experiments, for example, in tension and compression. We denote the tensile, compressive, and shear yield stresses (ultimate stresses) by σ_t, σ_c, and τ_0, respectively. Given the experimental values of σ_t and σ_c, from (1) we obtain

$$\beta = \sqrt{3}\,\frac{\sigma_c - \sigma_t}{\sigma_c + \sigma_t}, \qquad k = \frac{2}{\sqrt{3}}\,\frac{\sigma_c\sigma_t}{\sigma_c + \sigma_t}, \qquad \tau_0 = k, \qquad \sigma_0 = \frac{2}{3}\,\frac{\sigma_c\sigma_t}{\sigma_c - \sigma_t}. \tag{2}$$

To predict fracture direction with better accuracy, one should take into account that for the ultimate stress, the material characteristics k and β can differ from their values in the plastic region and depend on the normal stress σ.

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For plane stresses with $\sigma_2 = \sigma_y = 0$, the von Mises–Schleicher yield (strength) criterion is written in the principal stress axes as

$$\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2 = (\sqrt{3}k - \beta(\sigma_1 + \sigma_3)/\sqrt{3}\,)^2. \tag{3}$$

For convenience, we introduce auxiliary coordinates s and t (Fig. 1) related to the bisectrices of the first and second quadrants as follows:

$$s = (\sigma_1 + \sigma_3)/\sqrt{2}, \qquad t = (\sigma_3 - \sigma_1)/\sqrt{2}.$$

In these coordinates, condition (3) is simplified substantially:

$$(1 - 4\beta^2/3)s^2 + 3\,t^2 + 4\sqrt{2}\,k\beta s = 6k^2. \tag{4}$$

The adhesion k can be defined using (2) in terms of the ultimate tensile and compressive stresses (σ_t and σ_c) or σ_t and the friction coefficient β. Below, we define all strength parameters in terms of σ_t and β:

$$k = \tau_0 = \frac{\sqrt{3} + \beta}{3}\,\sigma_t, \qquad \sigma_c = \frac{\sqrt{3} + \beta}{\sqrt{3} - \beta}\,\sigma_t.$$

Equation (4) describes a second-order curve whose right vertex has the coordinates

$$\sigma_1 = \sigma_3 = \frac{\sqrt{3} + \beta}{\sqrt{3} + 2\beta}\,\sigma_t.$$

The shape of curve (4) is specified by the value of the internal-friction coefficient β. If $\beta \leqslant \sqrt{3}/2$, we have the equation of an ellipse

$$(s - s_0)/a^2 + t^2/b^2 = 1, \tag{5}$$

tilted at an angle of 45° to the axes σ_1 and σ_3 (see Fig. 1). The center and semiaxes of the ellipse are given by

$$s_0 = -\frac{2\sqrt{2}\beta(\sqrt{3} + \beta)}{3 - 4\beta^2}\,\sigma_t, \qquad a = \frac{\sqrt{6}\,(\sqrt{3} + \beta)}{3 - 4\beta^2}\,\sigma_t, \qquad b = \sqrt{\frac{2}{3}}\,\frac{\sqrt{3} + \beta}{\sqrt{3 - 4\beta^2}}\,\sigma_t.$$

For $\beta = 0$, we obtain the well-known Houber–von Mises ellipse with semiaxes $a = \sqrt{2}\,\sigma_t$ and $b = \sqrt{2/3}\,\sigma_t$ centered at the coordinate origin [9]. In Fig. 1, solid curves 2 and 3 show the von Mises and von Mises–Schleicher ellipses, respectively; here and below, dashed curve 1 shows the Coulomb–Mohr hexagon.

For $\beta = \sqrt{3}/2$, from (3) we obtain $\sigma_c = 3\sigma_t$. In this case, Eq. (4) is the equation of a parabola

$$t^2 + \sqrt{2}\,\sigma_t\, s = 3\sigma_t^2/2. \tag{6}$$

The coordinates of the parabola vertex determined for $t = 0$ are $\sigma_1 = \sigma_3 = 3\sigma_t/4$. Obviously, the vertex of the parabola is closer to the coordinate origin than the vertex of the ellipse, which is inconsistent with the Coulomb–Mohr criterion (Fig. 2).

For $\beta > \sqrt{3}/2$, Eq. (4) becomes the equation of a hyperbola

$$(s - s_0)/a^2 - t^2/b^2 = 1, \tag{7}$$

where

$$s_0 = \frac{2\sqrt{2}\,\beta(\sqrt{3} + \beta)}{4\beta^2 - 3}\,\sigma_t, \qquad a = \frac{\sqrt{6}\,(\sqrt{3} + \beta)}{4\beta^2 - 3}\,\sigma_t, \qquad b = \sqrt{\frac{2}{3}}\,\frac{\sqrt{3} + \beta}{\sqrt{4\beta^2 - 3}}\,\sigma_t.$$

The equations of the asymptotes of hyperbola (7) are given by

$$t = \pm(2/3)\sqrt{\beta^2 - 3/4}\,(s - s_0).$$

For $\beta = \sqrt{3}$, the coordinates of the hyperbola vertex are $\sigma_1 = \sigma_3 = 2\sigma_t/3$, the semiaxes are $a = b = 2\sqrt{2}\sigma_t/3$, and the center of the hyperbola lies at the center $\sigma_1 = \sigma_3 = 4\sigma_t/3$. The equations of the asymptotes become $\sigma_1 = 4\sigma_t/3$ and $\sigma_3 = 4\sigma_t/3$. In Fig. 2, the von Mises–Schleicher parabola and hyperbola are shown by solid curves 2 and 3, respectively.

We now consider Coffin's experimental data on gray cast iron [2] (Fig. 3) and those of Cornet and Grassi [2] on modified cast iron (Fig. 4). In Fig. 3, the open points refer to the results of Coffin and the solid curves refer to

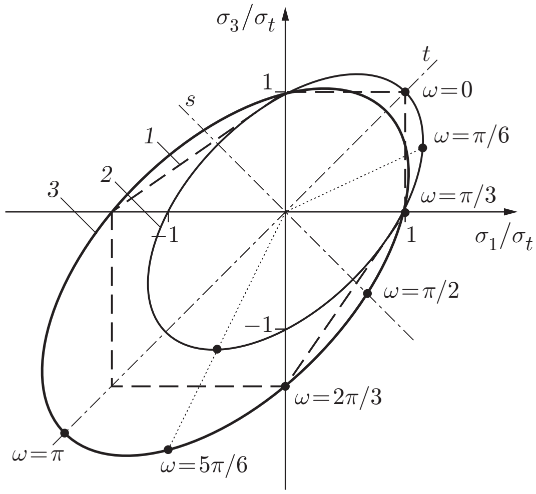

Fig. 1

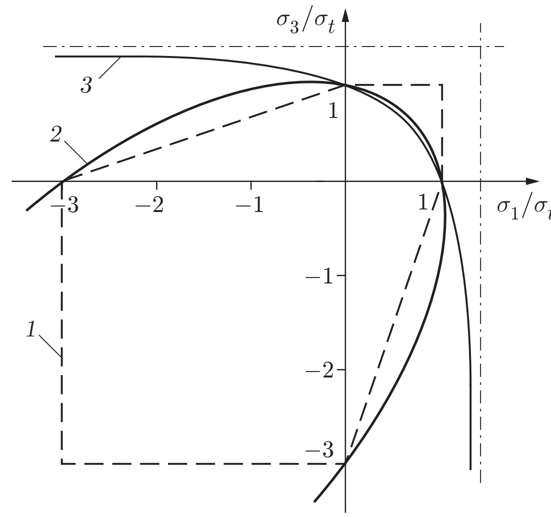

Fig. 2

Fig. 1. Von Mises–Schleicher ellipse in the plane of the stresses σ_1 and σ_3.

Fig. 2. Von Mises–Schleicher parabola and hyperbola in the plane σ_1, σ_3.

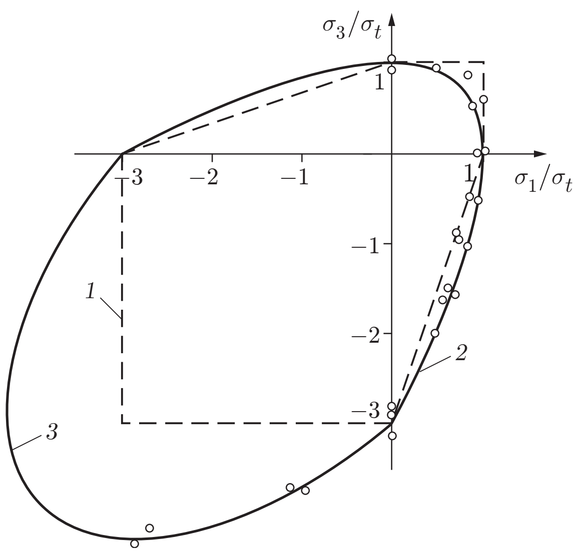

Fig. 3

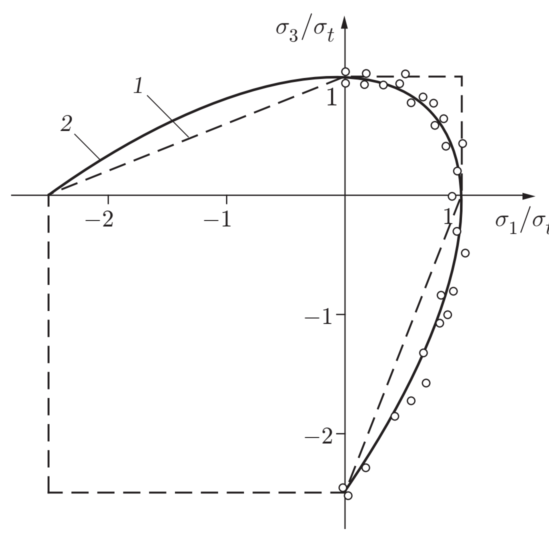

Fig. 4

Fig. 3. Comparison of calculation results with Coffin's experimental data for gray cast iron.

Fig. 4. Comparison of calculation results with the results of Cornet and Grassi for modified cast iron.

the von Mises–Schleicher criterion; in the first, second, and fourth quadrants, condition (4) is a parabola (curve 2) for $\beta = \sqrt{3}/2 \approx 0.866$ and $\sigma_c = 3\sigma_t$, and in the third quadrant, it is an ellipse (curve 3) for $\beta = 0.373$. Based on the experimental data of Coffin, we assume that for uniaxial compression, the quantity β is equal to the average value for the third and fourth quadrants; then $\beta = 0.62$. Below, this value of β is used to determine the direction of the compression fracture plane.

Cornet and Grassi performed experiments on specimens of gray and modified cast iron. The results for gray cast iron are close to the data of Coffin and are not given here. The experimental results for modified cast iron are shown in Fig. 4 and also agree well with condition (4) for $\beta = 0.742$.

The above results show that compared to the Coulomb–Mohr criterion, the von Mises–Schleicher criterion agrees better with experimental results on fracture stresses. This is most clearly seen from Fig. 3, which gives experimental data for the third quadrant. This important finding has not been reported in the literature, and the Coulomb–Mohr strength criterion, in its various modifications, [2] has been the most extensively studied and used in papers on plastic deformation and fracture of brittle materials.

Following [9], we introduce an angle ω that characterizes the type of stress state. Then, for the principal normal stresses, we can write

$$\sigma_1 = \sigma + \frac{2}{\sqrt{3}} T \cos\Big(\omega - \frac{\pi}{3}\Big), \quad \sigma_2 = \sigma - \frac{2}{\sqrt{3}} T \cos\omega, \quad \sigma_3 = \sigma + \frac{2}{\sqrt{3}} T \cos\Big(\omega + \frac{\pi}{3}\Big), \tag{8}$$

where ω is defined by the formula

$$\cos 3\omega = -(3\sqrt{3}\, I_3)/(2T^3), \qquad I_3 = s_x s_y s_z - s_x \tau_{yz}^2 - s_y \tau_{xz}^2 - s_y \tau_{xy}^2 + 2\tau_{xy}\tau_{xz}\tau_{yz}.$$

Here s_x, s_y, and s_z are the diagonal components of the stress deviator and $s_i = \sigma_i - \sigma$ $(i = x, y, z)$. Let us consider some types of stress state. For example, we have $\omega = \pi/6$ for biaxial tension $2\sigma_1 = \sigma_3$ (generalized shear), $\omega = \pi/3$ for tension, $\omega = \pi/2$ for pure shear $\sigma_1 = -\sigma_3$, and $\omega = 2\pi/3$ for compression.

For plane stresses (in direction 2), we obtain σ from (8) and then determine T from Eq. (1):

$$\sigma = \frac{2}{\sqrt{3}} T \cos\omega, \qquad T = \frac{\sqrt{3}\, k}{\sqrt{3} + 2\beta\cos\omega}. \tag{9}$$

With allowance for (9), formulas (8) become

$$\sigma_1 = 2T\cos(\omega - \pi/6), \qquad \sigma_3 = 2T\cos(\omega + \pi/6). \tag{10}$$

We assume that the y axis coincides with the second principal direction of the stress tensor and the x axis makes an angle θ with the first principal direction, for which $\tan 2\theta = 2\tau_{xz}/(\sigma_x - \sigma_z)$. Next, using (10) and the well-known formulas, we express the stresses in terms of the functions ω and θ in an arbitrary coordinate system:

$$\begin{pmatrix} \sigma_x \\ \sigma_z \end{pmatrix} = T(\sqrt{3}\cos\omega \pm \sin\omega\cos 2\theta), \qquad \tau_{xz} = T\sin\omega\sin 2\theta. \tag{11}$$

Replacing the expression for T in (11) by its value from (9), we obtain

$$\begin{pmatrix} \sigma_x \\ \sigma_z \end{pmatrix} = \frac{\sqrt{3}\, k(\sqrt{3}\, \cos\omega \pm \sin\omega\cos 2\theta)}{\sqrt{3} + 2\beta\cos\omega}, \qquad \tau_{xz} = \frac{\sqrt{3}\, k\sin\omega\sin 2\theta}{\sqrt{3} + 2\beta\cos\omega}.$$

Substituting σ_x, σ_z, and τ_{xz} into the equilibrium equation and differentiating, we obtain

$$\begin{gathered}\Big(\sqrt{3}\,\sin\omega\cos 2\theta - \cos\omega - \frac{2\beta}{\sqrt{3}}\Big)\frac{\partial\omega}{\partial x} + \sqrt{3}\sin\omega\sin 2\theta\,\frac{\partial\omega}{\partial z} - 2\sin\omega\Big(1 + \frac{2\beta}{\sqrt{3}}\,\cos\omega\Big)\frac{\partial\theta}{\partial z} = 0, \\ \sqrt{3}\,\sin\omega\sin 2\theta\,\frac{\partial\omega}{\partial x} - \Big(\sqrt{3}\,\sin\omega\cos 2\theta + \cos\omega + \frac{2\beta}{\sqrt{3}}\Big)\frac{\partial\omega}{\partial z} + 2\sin\omega\Big(1 + \frac{2\beta}{\sqrt{3}}\,\cos\omega\Big)\frac{\partial\theta}{\partial x} = 0.\end{gathered} \tag{12}$$

These partial differential equations for $\beta = 0$ are identical to the equations for plastic materials [9], and in the region of hyperbolicity of these equations, the characteristic lines are given by the equations

$$\frac{dz}{dx} = \tan\,(\theta - \psi), \qquad \frac{dz}{dx} = \tan\,(\theta + \psi), \tag{13}$$

where ψ is the angle between the first characteristic and the σ_1 axis:

$$\psi = \psi_\sigma = \frac{\pi}{2} - \frac{1}{2}\,\arccos\Big(\frac{\cot\,\omega}{\sqrt{3}} + \frac{2\beta}{3\sin\omega}\Big). \tag{14}$$

In the notation $\sin\varphi = \beta/\sqrt{3}$, the hyperbolicity condition for system (12) is written as

$$\cos^2\omega + \sin\varphi\cos\omega + \sin^2\varphi - 3/4 < 0. \tag{15}$$

Solving this inequality, we obtain

$$-\cos(\varphi - \pi/6) < \cos\omega < \cos(\varphi + \pi/6). \tag{16}$$

Generally, the angle φ varies from 0 to $\pi/2$, depending on brittleness of the material. We have $\varphi = \beta = 0$ for plastic metals, $\varphi \geqslant \pi/6$ for brittle materials, and $\varphi = \pi/2$ for cleavage failure. The hyperbolicity condition (16) takes the simplest form if $\varphi \leqslant \pi/6$ or $\varphi \geqslant \pi/6$.

For $\varphi \leqslant \pi/6$, inequality (16) implies

$$\varphi + \pi/6 < \omega < \varphi + 5\pi/6. \tag{17}$$

For $\varphi = \beta = 0$, inequality (17) implies the hyperbolicity condition [9] in the form

$$\pi/6 < \omega < 5\pi/6.$$

If the internal-friction angle φ increases from 0 to $\pi/6$, the right bound of the region of hyperbolicity increases to π and the left bound to $\pi/3$. As a result, for $\varphi = \pi/6$ ($\beta = \sqrt{3}/2$), the hyperbolicity condition becomes

$$\pi/3 \leqslant \omega < \pi.$$

We now consider brittle materials for which $\varphi \geqslant \pi/6$. In this case, (16) implies the inequality

$$\varphi + \pi/6 < \omega < 7\pi/6 - \varphi.$$

As the angle φ increases from $\pi/6$ to $\pi/2$, the right bound of this inequality decreases and the left bound increases to the value $\varphi = 2\pi/3$, and for $\varphi = \pi/2$ ($\beta = \sqrt{3}$) and any angles ω, the differential equations (12) are elliptic.

Let the functions $\omega = \omega(s)$ and $\theta = \theta(s)$ be specified along a certain line $x = x(s)$, $y = y(s)$. The solutions of the differential equations $\omega = \omega(x, z)$ and $\theta = \theta(x, z)$ form a certain surface (integral surface). The main question is whether a certain integral surface can be drawn through a given line L (Cauchy problem). For the integral surface passing through the line L, we write the obvious relations

$$\frac{\partial \omega}{\partial x}\, dx + \frac{\partial \omega}{\partial z}\, dz = d\omega, \qquad \frac{\partial \theta}{\partial x}\, dx + \frac{\partial \theta}{\partial z}\, dz = d\theta. \tag{18}$$

Along L, Eqs. (12) and (18) form a system on inhomogeneous linear algebraic equations for the first partial derivatives of the functions $\omega = \omega(x, z)$ and $\theta = \theta(x, z)$. If the line L is a characteristic of Eqs. (12), the derivatives are determined ambiguously along this line and, hence, the determinant of the above-mentioned algebraic system and the appropriate nominators in Cramer's formulas vanish. Equating the determinant of the system to zero, we arrive at the differential equations of the characteristic lines (13). Setting the nominators in Cramer's formula equal to zero, we obtain differential relations between unknown functions ω and θ that hold along the characteristics

$$\pm \frac{\sqrt{3}\, \Sigma(\omega)}{2 \sin \omega (\sqrt{3} + 2\beta \cos \omega)}\, d\omega - d\theta = 0, \tag{19}$$

where $\Sigma(\omega) = \sqrt{3 \sin^2 \omega - (\cos \omega + 2\beta/\sqrt{3})^2}$.

We introduce a new function λ using the equations

$$d\lambda = -\frac{\sqrt{3}\, \Sigma(\omega)}{2 \sin \omega (\sqrt{3} + 2\beta \cos \omega)}\, d\omega, \qquad \lambda = -\frac{\sqrt{3}}{2} \int\limits_{\omega_\beta}^{\omega} \frac{\Sigma(\omega)}{2 \sin \omega (\sqrt{3} + 2\beta \cos \omega)}\, d\omega. \tag{20}$$

In these relations, $\omega_\beta = \varphi + \pi/6$ and $\varphi = \arcsin(\beta/\sqrt{3})$. If $\beta = 0$, then $\omega_\beta = \pi/6$, which agrees with [9]; for $\beta = \sqrt{3}/2$, we obtain $\omega_\beta = \pi/3$. Thus, system (12) has two families of Characteristics, for which the following relations hold:

$$\frac{dz}{dx} = \tan\, (\theta - \psi), \qquad \theta - \lambda = \text{const} = \xi \qquad \text{along the first line,}$$

$$\frac{dz}{dx} = \tan\, (\theta + \psi), \qquad \theta + \lambda = \text{const} = \eta \qquad \text{along the second line.}$$

To derive and study the equations for the velocity field, we consider the dilatancy plastic model of [5, 6] whose determining relations are represented as the result of shears over a finite number of slip systems [10]. Below, we use the plane-stress relations of this model [10]:

$$\dot{e}_x = \Big(\frac{\Lambda}{3} + \frac{s_x}{2T}\Big)\dot{\Gamma}_p, \qquad \dot{e}_z = \Big(\frac{\Lambda}{3} + \frac{s_z}{2T}\Big)\dot{\Gamma}_p, \qquad \dot{\gamma}_{xz} = \frac{\tau_{xz}}{T}\, \dot{\Gamma}_p, \qquad \dot{e}_y = \Big(\frac{\Lambda}{3} + \frac{s_y}{2T}\Big)\dot{\Gamma}_p,$$

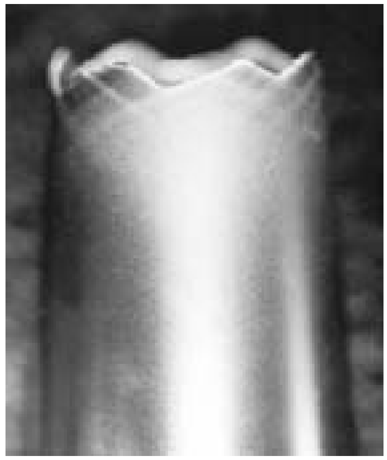

Fig. 5. Fracture directions in 12KhN3A steel specimens.

where Λ is the dilatancy coefficient and $\dot{\Gamma}_p$ is the intensity of the plastic shear strain rates. Eliminating the parameter $\dot{\Gamma}_p$ from these relations and substituting the stress-deviator components $s_x = T(\cos\omega/\sqrt{3} + \sin\omega\cos 2\theta)$ and $s_z = T(\cos\omega/\sqrt{3} - \sin\omega\cos 2\theta)$ derived from (11), we obtain the equations for the velocity-vector components v_x and v_z

$$\tan 2\theta\,\frac{\partial v_x}{\partial x} - \frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} - \tan 2\theta\,\frac{\partial v_z}{\partial z} = 0, \qquad (a\cos 2\theta - b)\,\frac{\partial v_x}{\partial x} + (a\cos 2\theta + b)\,\frac{\partial v_z}{\partial z} = 0. \tag{21}$$

Here $a = \sin\omega$ and $b = 2\Lambda/3 + \cos\omega/\sqrt{3}$.

In the notation $\sin\varphi_v = \Lambda/\sqrt{3}$, where φ_v is the dilatancy angle, the hyperbolicity condition for system (21) is written as

$$\cos^2\omega + \sin\varphi_v\cos\omega + \sin^2\varphi_v - 3/4 < 0. \tag{22}$$

One can see from (22) that the hyperbolicity condition for the velocity field is identical to the condition for stresses (15) and all subsequent inequalities for $\varphi_v = \varphi$.

The equations of the characteristics of system (21) are similar to Eqs. (13) in which one sets $2\psi = 2\psi_v = \pi - \arccos(b/a)$. Taking into account the results obtained previously, we write the expressions for the angles ψ_σ and ψ_v which determine the directions of the characteristics for the stress and velocity fields:

$$\psi_\sigma = \frac{\pi}{2} - \frac{1}{2}\arccos\Big(\frac{\cot\omega}{\sqrt{3}} + \frac{2\beta}{3\sin\omega}\Big), \qquad \psi_v = \frac{\pi}{2} - \frac{1}{2}\arccos\Big(\frac{\cot\omega}{\sqrt{3}} + \frac{2\Lambda}{3\sin\omega}\Big). \tag{23}$$

As follows from these formulas, for $\Lambda = \beta$ (in the case of plastic flow associated with a von Mises–Schleicher surface), the characteristics of the system of velocity equations are identical to the characteristics for stresses since $\psi_\sigma = \psi_v$. Setting $\Lambda = \beta = 0$, we arrive at the results for plastic metals [9]. Since $\omega = \pi/3$ for uniaxial tension, substitution of this value into (23) yields $\psi_\sigma = \psi_v \approx 54.7^\circ$, which agrees with experimental results for flat specimens [1] and thin-walled cylinders made of 12KhN3A steel [11[(Fig. 5).

For the values of v_x and v_z specified on the line L, as in the case of the m of equations for stresses, we supplement Eqs. (21) by the differential relations

$$\frac{\partial v_x}{\partial x}\,dx + \frac{\partial v_x}{\partial z}\,dz = dv_x, \qquad \frac{\partial v_z}{\partial x}\,dx + \frac{\partial v_z}{\partial z}\,dz = dv_z. \tag{24}$$

Along L, Eqs. (21) and (24) form a system of inhomogeneous linear algebraic equations for the first partial derivatives of the functions $v_x = v_x(x, z)$ and $v_z = v_z(x, z)$. If the line L is a characteristic of Eqs. (21), the derivatives are determined ambiguously along the line and, hence, the determinant of the above-mentioned algebraic system and the appropriate nominators in Cramer's formulas vanish. Equating the determinant of the system to zero, we arrive at the differential equations of the characteristics, which coincide with (13) for $\psi = \psi_v$. Setting the nominators equal to zero, we obtain differential relations between unknown functions v_x and v_z that hold along each characteristic:

$$dv_x\,dx + dv_z\,dz = 0. \tag{25}$$

Fig. 6. Fracture directions on limestone specimens.

Substitution of the equations of characteristics into this formula yields two relations for the velocities along each characteristic. Let us derive these relations for the projections of the velocity vector u and v onto the tangents to the characteristic lines of the first and second families. We denote the projections of the velocity onto the normals to the first and second characteristics by u_n and v_n, respectively. In view of the aforesaid, v_x and v_z are expressed in terms of u and u_n:

$$v_x = u\cos\theta_\alpha - u_n \sin\theta_\alpha, \qquad v_z = u\sin\theta_\alpha + u_n\cos\theta_\alpha. \tag{26}$$

Here $\theta_\alpha = \theta - \psi_v$ is the angle between the characteristic of the first family and the x axis. Similarly, v_x and v_z are expressed in terms of v and v_n:

$$v_x = v_n \sin\theta_\beta + v\cos\theta_\beta, \qquad v_z = -v_n\cos\theta_\beta + v\sin\theta_\beta. \tag{27}$$

Here $\theta_\beta = \theta_\alpha + 2\psi_v = \theta + \psi_v$ is the angle between the characteristic of the second family and the x axis. Using formulas (25)–(27) along each characteristic, we obtain

$$du - u_n\,d\theta_\alpha = 0, \qquad dv + v_n\,d\theta_\alpha = 0. \tag{28}$$

Equating the right sides of formulas (26) and (27), and solving the resulting system of equations for u_n and v_n we obtain

$$u_n = v\mathrm{cosec}2\psi - u\cot 2\psi, \qquad v_n = u\mathrm{cosec}2\psi - v\cot 2\psi. \tag{29}$$

Substitution of these values into (28) yields the following relations for the velocity field on the characteristics:

$$\begin{aligned} &du - (v\mathrm{cosec}2\psi - u\cot 2\psi)\,d\theta_\alpha = 0 \qquad \text{along the line } \alpha,\\ &dv + (u\mathrm{cosec}2\psi - v\cot 2\psi)\,d\theta_\alpha = 0 \qquad \text{along the line } \beta. \end{aligned} \tag{30}$$

Let us consider some particular cases. For $2\psi = \pi/2$, Eqs. (30) become Geiringer's relations for plane strains of a rigid-plastic medium [9]; for $2\psi = \pi/2 + \varphi$, where φ is the internal-friction angle, the equations for the velocity components on the characteristics (30) become

$$du - (v\sec\varphi + u\tan\varphi)\,d\theta_\alpha = 0 \qquad \text{along the line } \alpha,$$

$$dv + (u\sec\varphi + v\tan\varphi)\,d\theta_\alpha = 0 \qquad \text{along the line } \beta.$$

Shield [12] obtained these relations for the velocity field of a Coulomb–Mohr ideal rigid-plastic soil under plane-strain conditions. The relations for plane stresses of a Levy–von Mises rigid-plastic incompressible material [13] are another particular case of the equations on characteristics (30).

As noted above, $\omega = 2\pi/3$ for uniaxial compression. In this case, using Coffin's experimental data, we obtain $\beta = 0.62$. Setting $\Lambda = \beta$ and substituting these values into (23), we obtain $\psi_\sigma = \psi_v \approx 49°$. These results are in good agreement with the experiments of [14], in which cylindrical specimens of gray cast iron failed at an angle of approximately 45°.

It was found experimentally [2, 15] that compressive fracture of brittle rocks in the absence of friction at the ends occurs over planes parallel to the compression direction, i.e., when $\psi_v = \pi/2$. This result follows from (23) for $\Lambda = \sqrt{3}$. Figure 6 shows results of our previous experiments (at the Institute of Mining Science, Russian Academy of Sciences) with cylindrical limestone specimens whose ends were lubricated with paraffin wax.

The above comparison of experimental and theoretical results for plastic and brittle solids shows that the von Mises–Schleicher criterion adequately predicts the limiting stresses and fracture directions identified with the velocity-field characteristics.

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